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INTEGRO-DIFFERENTIAL EQUATIONS OF THE PROBLEM OF THE SCATTERING OF ELASTIC WAVES BY A PLANE THIN-WALLED INCLUSION OF LARGE STIFFNESS†

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Integro-differential equations of the problem of the scattering of elastic waves by a thin-walled plane inclusion of large stiffness are presented on the assumption that the scatterer and the matrix are rigidly coupled. The problem of the scattering of a longitudinal wave by a tunnel inclusion is considered as an example. © 1996 Elsevier Science Ltd. All rights reserved.

1. We will consider an elastic uniform medium characterized by Lamé parameters λ and μ and a density ρ, in which, assuming rigid contact, there is a foreign elastic inclusion which occupies the region $\{(x_1, x_2) \in S, |x_3| < h/2\}$, where S is its middle surface, bounded by a closed smooth contour ∂S with outward normal $\mathbf{n}$, h is the thickness, and x_1, x_2 and x_3 are Cartesian coordinates. The material of the inclusion has the parameters λ_0, μ_0, ρ_0.

We will discuss the case when μ_0/μ, $\lambda_0/\lambda \gg 1$, $h \to 0$. Here μ_0/μ, λ_0/λ approach infinity as $(h/a)^{-\kappa}$, $\kappa > 0$, where a is the characteristic dimension of the region S. Clearly, as $h \to 0$ the equations of motion of the particles of the inclusion (Lamé's equations) reduce to the corresponding equations of motion of an elastic plate. The type of oscillations of the latter due to the action of a travelling elastic wave will be determined by the value of the parameter κ. Thus [1], when $\kappa > 3$ the boundary elastic inclusion can be modelled by an absolutely rigid inclusion, and when $\kappa = 1, 3$ they follow the equations of longitudinal (symmetric) or transverse (flexural, antisymmetric) oscillations of an elastic plate, respectively. Then, naturally, the frequency band of the oscillations in governed by the condition $k_T h < 1$, where $k_A = \omega/c_A$ is the wave number of the longitudinal bulk waves of the external medium ($A = L$) or the transverse bulk waves ($A = T$), and ω is the angular frequency.

Suppose D is a certain closed doubly connected region in R^3, $\mathbf{F} = (F_1, F_2, F_3)$ is the density of volume forces acting in D, $e_{ij}(\mathbf{u})$ $(i, j = 1, 2, 3)$ is the strain tensor, corresponding to the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$, and $\partial D = S_0 \cup \{(x_1, x_2) \in S, |x_3| = h/2\}$ is the boundary of the region D, where the external contour S_0 with normal $\mathbf{n}^0$ is assumed to be free of forces. We will introduce the following functionals

$$L(\mathbf{v}) = \int_D (F_i + \rho\omega^2 u_i)\upsilon_i d\mathbf{x}, \quad d\mathbf{x} = dx_1 dx_2 dx_3$$

$$A(\mathbf{u}, \mathbf{v}) = \int_D [\lambda e_{ii}(\mathbf{u}) e_{jj}(\mathbf{v}) + 2\mu\, e_{ij}(\mathbf{u}) e_{ij}(\mathbf{v})] d\mathbf{x}, \quad i, j = 1, 2, 3$$

corresponding to the work done by the external inertial and internal forces along the virtual displacements $\mathbf{v} = (\upsilon_1, \upsilon_2, \upsilon_3)$. In these formulae and henceforth summation is carried out over repeated subscripts, the Latin subscripts take values of 1, 2 and 3, while the Greek subscripts take values of 1 and 2.

Then, when $\bar{\mu} \in R^+ \backslash \{0\}$, $\bar{\lambda} \in R^+ \cup \{\infty\}$, the vector of displacements $\mu \in \bar{V}_\kappa$ for all $\upsilon \in V_\kappa$ satisfies the relation $J_\kappa = 0$ $(\kappa = 1, 2, 3)$, apart from the displacement of the inclusion as a rigid whole. Here

$$J_1 = J_0 - \int_S [\lambda^* e_{\alpha\alpha}(\mathbf{u}) e_{\beta\beta}(\mathbf{v}) + 2\bar{\mu}\, e_{\alpha\beta}(\mathbf{u}) e_{\alpha\beta}(\mathbf{v})] dS + m_0 \int_S u_i \upsilon_i dS$$

as

$$\mu_0 h \to \bar{\mu}, \quad \lambda_0 h \to \bar{\lambda}, \quad h \to 0; \quad V_1 = \{\mathbf{v} \in H^1(D); \;\; \upsilon_\alpha|_S \in H^1(S)\}$$

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$$J_2 = J_0 + m_0 \int_S u_3 \upsilon_3 dS \tag{1.1}$$

as

$$\mu_0 h^2 \to \bar{\mu}, \quad \lambda_0 h^2 \to \bar{\lambda}, \quad h \to 0; \quad V_2 = \{\boldsymbol{\upsilon} \in H^1(D); \ \upsilon_\alpha|_S = 0\}$$

$$J_3 = J_0 - \frac{1}{12} \int_S [\lambda^* u_{3,\alpha\alpha} \upsilon_{3,\beta\beta} + 2\bar{\mu} u_{3,\alpha\beta} \upsilon_{3,\alpha\beta}] dS + m_0 \int_S u_3 \upsilon_3 dS$$

as

$$\mu_0 h^3 \to \bar{\mu}, \quad \lambda_0 h^3 \to \bar{\lambda}, \quad h \to 0; \quad V_3 = \{\boldsymbol{\upsilon} \in V_2, \ \upsilon_3 \in H^2(S)\}$$

and

$$J_0 = L(\boldsymbol{\upsilon}) - A(\mathbf{u}, \boldsymbol{\upsilon}), \quad \lambda^* = \frac{2\bar{\lambda}\bar{\mu}}{(\bar{\lambda} + 2\bar{\mu})}, \quad dS = dx_1 dx_2, \quad u_{,\alpha\beta} = \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}, \quad m_0 = \rho_0 \omega^2 h$$

where the integrals over the surface S are evaluated at $x_3 = 0$ and H^m is the Hilbert space of functions having derivative up to order m, summable with a square, on S.

Expressions for the functionals (1.1) follow from the well-known results obtained in [1], if we take into account the inertial forces using d'Alembert's principle. The conditions for the functionals J_κ; $\partial J_\kappa = 0$ ($\kappa = 1, 2, 3$) to be stationary, using Green's formulae, which relate the integrals over a volume to the integrals over a surface and which convert integrals over the area of the middle surface S into integrals over the contour ∂S bounding it, can be written, respectively, in the form

$$\delta J_0 + \int_S \sigma^*_{\alpha\beta,\beta} \delta\upsilon_\alpha dS - \int_{\partial S} \sigma^*_{\alpha\beta} n_\beta \delta\upsilon_\alpha dl + m_0 \int_S u_i \delta\upsilon_i dS = 0, \quad \kappa = 1$$

$$\delta J_0 + m_0 \int_S u_3 \delta\upsilon_3 dS = 0, \quad \kappa = 2$$

$$\begin{aligned} &\delta J_0 - \frac{1}{12}\left\{(\lambda^* + \bar{\mu}) \int_S \nabla^4 u_3 \delta\upsilon_3 dS + (\lambda^* + 2\bar{\mu}) \int_{\partial S} \left(u_{3,\alpha\alpha} \frac{\partial}{\partial n} \delta\upsilon_3 - \delta\upsilon_3 \frac{\partial}{\partial n} u_{3,\alpha\alpha}\right) dL + \right. \\ &\left. + 2\bar{\mu} \int_{\partial S} (u_{3,11}\delta\upsilon_{3,2} - u_{3,12}\delta\upsilon_{3,1}) dx_1 + (u_{3,12}\delta\upsilon_{3,2} - u_{3,22}\delta\upsilon_{3,1}) dx_2 \right\} + m_0 \int_S u_3 \delta\upsilon_3 dS = 0, \quad \kappa = 3 \end{aligned} \tag{1.2}$$

$$\delta J_0 = \int_D (\sigma_{ij,j} + \rho\omega^2 u_i + F_i) \delta\upsilon_i d\mathbf{x} - \int_{\partial D} \sigma_{ij} n_j \delta\upsilon_i dA$$

$$\sigma^*_{\alpha\beta} = \lambda^* u_{\alpha,\alpha} \delta_{\alpha\beta} + 2\bar{\mu} e_{\alpha\beta}, \quad \nabla^4 = \nabla^2\nabla^2, \quad \nabla^2 u = u_{,\alpha\alpha}$$

where dA is an element of area of the surface ∂D, dl is an element of the arc of the positive direction of the contour ∂S, σ_{ij} is the stress tensor corresponding to the tensor e_{ij} and δ_{ij} is the Kronecker delta.

In view of the fact that the variations $\partial\delta\upsilon_3/\partial n$, $\delta\upsilon_i$ and the condition $\sigma_{ij} n_j^0 = 0$ are arbitrary on S_0, from relations (1.2) when $\mathbf{x} \in R^3 \backslash S$ we obtain the equations of steady Lamé oscillations, which, ignoring mass forces, we will write in the form

$$(\lambda + \mu) u_{i,ij} + \mu u_{j,ii} + \rho\omega^2 u_j = 0 \tag{1.3}$$

while when $\mathbf{x} \in S$ we obtain the equations of motion for the thin-walled inclusion corresponding to the boundary conditions on its contour

$$\Phi_\beta = -E_1 h(\Delta^* + k_l^2) u_\beta, \quad \Phi_3 = -E_1 h k_l^2 u_3, \quad \mathbf{x} \in S \tag{1.4}$$

$$\sigma^*_{\alpha\beta} n_\beta = 0, \quad \mathbf{x} \in \partial S; \quad \kappa = 1$$

$$u_1 = u_2 = 0, \quad \Phi_3 = -E_1 h k_l^2 u_3, \quad \mathbf{x} \in S; \quad \kappa = 2 \tag{1.5}$$

$$u_1 = u_2 = 0, \quad \Phi_3 = g(\nabla^4 - k_b^4)u_3, \quad \mathbf{x} \in S \tag{1.6}$$

$$\frac{\partial}{\partial n}\nabla^2 u_3 + \frac{\partial}{\partial \tau} M_t u_3 = 0, \quad M_n u_3 = 0, \quad \mathbf{x} \in \partial S, \quad \kappa = 3$$

Here

$$E_1 = \frac{E_0}{1-\nu_0^2}, \quad g = \frac{E_1 h^3}{12}, \quad c_l = \left(\frac{E_1}{\rho_0}\right)^{1/2}, \quad c_b = \left(\frac{\omega^2 g}{\rho_0 h}\right)^{1/4}$$

$$k_f = \frac{\omega}{c_f}\ (f = l, b), \quad \Phi_i = \sigma_{i3}^+ - \sigma_{i3}^- \quad (\sigma_{i3}^\pm = \lim_{h\to 0} \sigma_{i3}(x_1, x_2, \pm h))$$

$$M_1 = u_{3,11} + \nu_0 u_{3,22}, \quad M_2 = u_{3,22} + \nu_0 u_{3,12}, \quad M_{12} = (1-\nu_0)u_{3,12}$$

$$M_t u_3 = (M_2 - M_1)n_1 n_2 + M_{12}(n_1^2 - n_2^2), \quad M_n u_3 = M_1 n_1^2 = M_2 n_2^2 + 2M_{12} n_1 n_2$$

$$\Delta^* u_\beta = ((1+\nu_0)u_{\alpha,\alpha\beta} + (1-\nu_0)u_{\beta,\alpha\alpha})/2$$

(E_1 is the modulus of elasticity for a thin plate, c_1 is the velocity of a longitudinal wave in the plate, c_b is the velocity of flexural waves in the plate, g is the cylindrical stiffness of the plate, Φ_i is the jump in the stresses, E_0 and ν_0 are Young's modulus and Poisson's ratio of the material of the inclusion and τ is the unit vector of the tangent to ∂S, obtained from $\mathbf{n}$ by rotation by $+\pi/2$). The boundary conditions of problems (1.4) and (1.6) are the conditions for the edge of the plate not to be fixed (free).

Equations (1.5) and (1.6) were obtained ignoring effects due to transverse compression oscillations of the inclusion and are asymptotically correct if the frequency of the incident wave is not the same as the natural frequency of the homogeneous internal boundary-value problem, defined by relations (1.4). To investigate the situation when these two frequencies are the same and to obtain equations which combine the cases $\kappa = 1, 2, 3$, we use the method of matched asymptotic expansions [2–4]. As a result we obtain the following relations

$$\Phi_\beta = -E_1 h\ (\Delta^* + k_l^2)u_\beta, \quad \mathbf{x} \in S; \quad \sigma_{\alpha\beta}^* n_\beta = 0, \quad \mathbf{x} \in \partial S \tag{1.7}$$

$$\Phi_3 = g(\nabla^4 - k_b^4)u_3, \quad \mathbf{x} \in S; \quad \frac{\partial}{\partial n}\nabla^2 u_3 + \frac{\partial}{\partial \tau} M_t u_3 = 0, \quad M_n u_3 = 0, \quad \mathbf{x} \in \partial S \tag{1.8}$$

which describe, asymptotically exactly, the interaction between the elastic medium and a thin plane inclusion of large stiffness when the surface of inhomogeneity and the matrix are rigidly coupled. (This assertion was proved previously by Ya. I. Kunts in an unpublished paper.)

2. Suppose that an elastic wave, characterized by a displacement vector $u_i(x)$ (the time factor $\exp(-i\omega t)$ is assumed) is incident on the inclusion considered. Then, outside the surface of the obstacle, the field $\mathbf{u}(\mathbf{x}) = \mathbf{u}^i(\mathbf{x}) + \mathbf{u}^s(\mathbf{x})$ satisfies the equation of steady Lamé oscillations (1.3), while in the region $\mathbf{x} \in S$ it satisfies relations (1.7) and (1.8). The scattered field $u_s(x)$ then satisfies the Sommerfeld radiation condition, from which it follows that

$$\mathbf{u}^s(\mathbf{x}) \simeq -\frac{1}{4\pi R} \sum_{A=L,T} \exp(ik_A R)\mathbf{f}^A(\omega, \boldsymbol{\nu}) \quad (R = |\mathbf{x}| \to \infty) \tag{2.1}$$

where $\mathbf{f}^A(\omega; \boldsymbol{\nu})$ is the vector amplitude of the scattering of longitudinal waves ($A = L$) and transverse waves ($A = T$), and $\boldsymbol{\nu} = \mathbf{x}/R$ is the direction of observation.

From Betti's reciprocity theorem for an infinite region [5], taking into account the condition $\mathbf{u}(\mathbf{x}) \simeq \mathbf{u}(x_1, x_2)$ when $|x_3| < h/2$, we have

$$\mathbf{u}^s(\mathbf{x}) = -\int_S \Phi_i(\mathbf{y})\mathbf{G}_i(\mathbf{x},\mathbf{y})d\mathbf{y}, \quad d\mathbf{y} = dy_1 dy_2, \quad x_3 > 0; \quad \mathbf{G}_i = (G_{1i}, G_{2i}, G_{3i}) \tag{2.2}$$

$$G_{ij} = \frac{g_T}{\mu}\delta_{ij} - \frac{1}{\rho\omega^2}\frac{\partial^2(g_L - g_T)}{\partial y_i \partial y_j}, \quad g_A = \frac{\exp(ik_A|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|} \quad (A = L, T)$$

We obtain the following expressions for the scattering amplitudes from (2.1) and (2.2)

$$\mathbf{f}^A(\omega,\boldsymbol{\nu}) = -\frac{1}{\mu}\mathbf{a}_i^A \int_S \Phi_i(\mathbf{y})\exp[-ik_A(\boldsymbol{\nu},\mathbf{y})]d\mathbf{y} \tag{2.3}$$

$$\mathbf{a}_i^L = \xi^2\boldsymbol{\nu}\nu_i, \quad \mathbf{a}_i^T = \boldsymbol{\delta}_i - \boldsymbol{\nu}\nu_i, \quad \boldsymbol{\delta}_i = (\delta_{1i},\delta_{2i},\delta_{3i}), \quad \xi = c_T / c_L$$

Using the expansion of a spherical wave in plane waves, relation (2.2) can be written in the following form when $x_3 = 0$

$$\begin{aligned}
&u_\beta(x_1,x_2) = u_\beta^i(x_1,x_2) - \frac{\xi^2}{2\mu}k_L\int_S \Phi_\gamma(\mathbf{y})K_{\gamma\beta}(k_L|\mathbf{r}|)d\mathbf{y}\\
&u_3(x_1,x_2) = u_3^i(x_1,x_2) + \frac{\xi^2}{2\mu}k_L\int_S \Phi_3(\mathbf{y})K_3(k_L|\mathbf{r}|)d\mathbf{y}\\
&K_{\gamma\beta}(|\mathbf{r}|) = \frac{1}{4\pi^2}\int_{R^2} L_{\gamma\beta}(\boldsymbol{\alpha})e^{i(\boldsymbol{\alpha},\mathbf{r})}d\boldsymbol{\alpha}, \quad L_{\gamma\beta} = \frac{\xi^{-2}}{\gamma_3}\delta_{\gamma\beta} + \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_3}\right)\alpha_\gamma\alpha_\beta\\
&K_3(|\mathbf{r}|) = \frac{1}{4\pi^2}\int_{R^2} L_3(\boldsymbol{\alpha})e^{i(\boldsymbol{\alpha},\mathbf{r})}d\boldsymbol{\alpha}, \quad L_3 = \gamma_1 - \frac{|\boldsymbol{\alpha}|^2}{\gamma_3}\\
&\mathbf{r} = (x_1 - y_1,\ x_2 - y_2), \quad d\boldsymbol{\alpha} = d\alpha_1 d\alpha_2, \quad \gamma_1 = (|\boldsymbol{\alpha}|^2 - 1)^{1/2}, \quad \gamma_3 = (|\boldsymbol{\alpha}|^2 - \xi^{-2})^{1/2}
\end{aligned} \tag{2.4}$$

where the branches of the radicals γ_i ($i = 1, 3$) are defined by the condition $\operatorname{Im}\gamma_i < 0$ for $|\boldsymbol{\alpha}| < 1$ and $|\boldsymbol{\alpha}| < \xi^{-1}$, respectively. The kernel of the integral representations (2.4) $K_{\gamma\beta}$, K_3 are polar (weakly singular) and possess the asymptotic form const $|\mathbf{r}|^{-1}$ as $|\mathbf{r}| \to 0$. Note that when $\mathbf{u}(\mathbf{x}) = 0$ on S we obtain from (2.4) integral equations corresponding to the problem of the scattering of an elastic wave by an absolutely rigid inclusion.

Hence, we arrive at systems of equations (1.7), (1.8) and (2.3), (2.4) which define the solution of the problem of the scattering of an elastic wave by a thin-walled elastic inclusion of large stiffness (i.e. provided that the wave impedance of the material of the inclusion is much larger than the wave impedance of the material of the medium). Uniform boundary-value problems (1.7) and (1.8) are self-conjugate (Hermitian), their eigenvalues are non-negative, and the eigenfunctions (they can be chosen to be real) are orthogonal and form a complete system in $H^0(S)$. (These assertions follow from the fact that the operators defined by the right-hand sides of Eqs (1.7) and (1.8) are positive definite and also from the conditions on the contour ∂S.)

Hence, the solution of problems (1.7) and (1.8) can be represented in the form of expansions in eigenfunctions and eigenvalues. Substitution of these expansions into (2.4) leads to integral equations with compact operators, acting from the space $H^0(S)$ into $C(S)$, to determine the required jumps in the stresses Φ_i. For the latter we will use all the statements of Fredholm's theory, and the self-regulation methods holds for the numerical solution of the integral equations of the first kind obtained in this way [6]. However, it is difficult to construct the eigenfunctions and eigenvalues for an arbitrary region S, except for canonical regions (a section and a circle).

On the other hand, assuming that the numbers k_e and k_b are not eigenvalues of the internal problems (1.7) and (1.8), the systems of equations considered can be reduced to corresponding integro-differential equations. In fact, substituting (2.4) into (1.7) and (1.8) we obtain

$$\Phi_\beta(x_1,x_2) - Zk_L^2(k_l^{-2}\Delta^* + 1)\int_S \Phi_\gamma(\mathbf{y})K_{\gamma\beta}(k_L|\mathbf{r}|)d\mathbf{y} = -E_1h(\Delta^* + k_l^2)u_\beta^i(x_1,x_2) \tag{2.5}$$

$$\Phi_3(x_1,x_2) - Zk_L^2(k_b^{-2}\nabla^4 - 1)\int_S \Phi_3(\mathbf{y})K_3(k_L|\mathbf{r}|)d\mathbf{y} =$$

$$= g(\nabla^4 - k_b^4)u_3^i(x_1,x_2), \quad (x_1,x_2) \in S \tag{2.6}$$

$$\int_S \Phi_\beta(\mathbf{y})d\mathbf{y} - Zk_L^2\int_S\int_S \Phi_\beta(\mathbf{y})K_{\gamma\beta}(k_L|\mathbf{r}|)dS\,d\mathbf{y} = -m_0\int_S u_\beta^i(\mathbf{y})d\mathbf{y} \tag{2.7}$$

$$\int_S \Phi_3(y)dy - Zk_L^2 \iint_{SS} \Phi_3(y)K_3(k_L|\mathbf{r}|)dS\,dy = -m_0 \int_S u_3^i(y)dy \tag{2.8}$$

$$Zk_L^2 M_n \int_S \Phi_3(y)K_3(k_L|\mathbf{r}|)dy = -m_0' M_n u_3^i(x_1,x_2), \quad (x_1,x_2)\in\partial S \tag{2.9}$$

$$Z = k_L h \rho_0 / (2\rho)$$

Equations (2.7) and (2.8) are obtained by integrating the corresponding differential equations over S taking the condition on the contour ∂S into account.

3. Consider the case when the right-hand sides of Eqs (2.5) and (2.7) or (2.6), (2.8) and (2.9) are zero. Suppose $\phi_k(x_1, x_2)$ ($k = 1, 2, \dots, r$) are linearly independent eigenfunctions of the differential operators or problems (1.6) and (1.7), regular inside S. We will represent the general solution of the corresponding equations in the form

$$u_i(x_1,x_2) = \sum_{k=1}^{r} c_{ik}\phi_k(x_1,x_2) + \int_S \Phi_j(y)\Gamma_{ij}(|\mathbf{r}|)dy, \tag{3.1}$$

$$\Gamma_{\alpha\beta} = \frac{g_a}{\mu_0}\delta_{\alpha\beta} - \frac{1}{\rho_0\omega^2}\frac{\partial^2(g_a - g_d)}{\partial y_\alpha \delta y_\beta}, \quad g_f = \frac{1}{4h}N_0(k_f|\mathbf{r}|) \quad (f = a,d)$$

$$k_f = \frac{\omega}{c_f}, \quad c_a^2 = \frac{1+\nu_0}{2}c_l^2, \quad c_d^2 = \frac{1-\nu_0}{2}c_l^2$$

$$\Gamma_{3i} = -\frac{\delta_{i3}}{8m_0}\left[N_0(k_b|\mathbf{r}|) + \frac{2}{\pi}K_0(k_b|\mathbf{r}|)\right]$$

where c_{ik} are arbitrary constants, the functions ϕ_k are chosen to be real, and N_0 and K_0 are Neumann and MacDonald functions, respectively. Suppose, further, that the numbers k_l and k_b are the eigenvalues of boundary-value problems (1.7) or (1.8). Since these problems are elliptic boundary-value problems (it is easy to verify that the Shapiro–Lopatinskii condition holds for them), then for these to be solvable it is necessary and sufficient that [7]

$$\int_S \phi_k(y)\Phi_j(y)dy = 0, \quad k = 1,\dots,r \tag{3.2}$$

Note that when S is a section, problem (1.7) is the Sturm–Liouville problem and, consequently, $r = 1$, while for problem (1.8), $r = 2$.

Then, instead of the integro-differential equation (2.5) or (2.6) we obtain from (2.4) and (3.1) corresponding integral equations of the first kind of the Fredholm type

$$\int_S M_{ij}(|\mathbf{r}|)\Phi_i(y)dy + \sum_{k=1}^{r} c_{ik}\phi_k(x_1,x_2) = u_i^i(x_1,x_2), \quad (x_1,x_2)\in S$$

$$M_{\gamma\beta} = \Gamma_{\beta\gamma}(|\mathbf{r}|) + \frac{\xi^2}{2\mu}k_L K_{\gamma\beta}(k_L|\mathbf{r}|), \quad M_{i3} = \Gamma_{i3}(|\mathbf{r}|) - \delta_{i3}\frac{\xi^2}{2\mu}k_L K_3(k_L|\mathbf{r}|)$$

where the required functions Φ_i satisfy conditions (3.2).

4. As an example we will consider the case when $S = \{|x_1| < a, |x_2| < \infty\}$ and the incident wave is represented in the form $\mathbf{u}^i(\mathbf{x}) = \mathbf{l}\exp[ik_L(\mathbf{l},\mathbf{x})]$, $\mathbf{l} = (\sin\theta_0, 0, -\cos\theta_0)$ (here and henceforth we consider waves with vertical polarization: $u_2(\mathbf{x}) = 0$). We have a plane problem, where no quantities that characterize the wave depend on the coordinate x_2. The solutions of boundary-value problems (1.7) and (1.8) in this case have the form

$$u_\beta(t) = -\frac{\xi_\beta^{-1}}{4m_0}x\int_{-1}^{1}\Phi_\beta(u)g_\gamma(u,t;\ x\xi_\gamma^{-1})\delta_{\gamma\beta}du, \quad \gamma,\beta = 1,3 \tag{4.1}$$

$$g_1(u,t;\ x) = 2\{\cos[x(2-|t-u|)] + \cos[x|t+u|]\}/\sin 2x$$

$$g_3(u,t;\ x) = \sin[x|t-u|) + \exp[x|t-u|) + f_1\,\mathrm{ch}(xt) + f_2\,\mathrm{sh}(xt) + f_3\cos(xt) + f_4\sin(xt)$$

$$f_1 = \{\cos(xu) + e^{-x}\mathrm{ch}(xu)d_2^-(x)\}/d_1^+(x)$$

$$f_2 = \{\sin(xu) + e^{-x}\mathrm{sh}(xu)d_2^+(x)\} / d_1^-(x)$$
$$f_3 = \{\mathrm{ch}(xu) + \cos(xu)d_3(x)\} / d_1^+(x)$$
$$f_4 = \{sh(xu) - \sin(xu)d_3(x)\} / d_1^-(x)$$
$$d_1^\pm(x) = \mathrm{ch}\,x \sin x \pm \mathrm{sh}\,x \cos x, \quad d_2^\pm = \cos x \pm \sin x$$
$$d_3(x) = \mathrm{ch}\,x \cos x - \mathrm{sh}\,x \sin x$$
$$x = k_L a, \quad \xi_1 = c_l / c_L, \quad \xi_3 = c_b / c_L, \quad t = x_1 / a$$

Note that when $\sin(2x\xi_1^{-1}) = 0$ or $d_1^+(x\xi_3^{-1})d_1^-(x\xi_3^{-1}) = 0$ solutions corresponding to (3.1) and (3.2) in the two-dimensional case follow from (4.1). Substituting (4.1) into (2.4) and integrating with respect to y_2 first, we obtain integral equations of the first kind with a logarithmic singularity in the kernels

$$\frac{xZ}{m_0}\int_{-1}^{1} \Phi_\beta(u) M_\gamma(u,t)\delta_{\gamma\beta} du = -l_\beta \exp(i\omega l_1 t), \quad |t| < 1$$
$$M_\beta(u,t) = K_\beta(x|t-u|) + Z^{-1}\xi_\beta^{-1} g_\beta(u,t;\ x\xi_\beta^{-1})$$
$$K_1(z) = -\frac{i}{2}\left[H_0^{(1)}(z) + \frac{q(z)}{z}\right], \quad q(z) = \frac{1}{\xi} H_1^{(1)}\left(\frac{z}{\xi}\right) - H_1^{(1)}(z)$$
$$K_3(z) = -\frac{i}{2}\left[\frac{1}{\xi^2} H_0^{(1)}\left(\frac{z}{\xi}\right) - \frac{q(z)}{z}\right], \quad \xi = \frac{c_T}{c_L}$$

where $H_m^{(1)}$ are cylindrical Hankel functions. Here, when $R_0 = (x_1^2 + x_3^2)^{1/2} \to \infty$ the scattered field $\mathbf{u}^s(\mathbf{x})$ has the asymptotic representation

$$\mathbf{u}^s(\mathbf{x}) = \sum_{A=L,T} (8\pi k_A R_0)^{-1/2} \exp(ik_A R_0 + i\pi/4)\mathbf{f}^A(\omega; \mathbf{l}, \boldsymbol{\nu}) \tag{4.2}$$

$$\mathbf{f}^A(\omega; \mathbf{l}, \boldsymbol{\nu}) = -\frac{a}{\mu}\mathbf{a}_\beta^A \int_{-1}^{1} \Phi_\beta(u)\exp(-ik_A a \nu_1 u) du, \quad \beta = 1,3, \quad A = L,T$$

We will choose as the scattering characteristics the total transverse scattering cross-section, defined in the directions $\theta_0 = 0, \pi/2$ by the equation $\sigma(\theta_0) = k_L^{-1}\mathrm{Im}(\mathbf{l}, \mathbf{f}^L(\omega: \mathbf{l}, \mathbf{l}))$, $(\mathbf{l}, \mathbf{a}_\beta^T) = 0$, and the polar scattering characteristic in the locational direction $F(\theta_0) = |\,\mathbf{f}^A(\omega: \mathbf{l}, -\mathbf{l})\,|$. The solution of the system of integral equations (4.2) was constructed numerically by the method of mechanical quadratures [6]. The results obtained were compared with the results by solving integro-differential equations (2.5)–(2.9) by the same method, written for the section, and were confirmed by them. Here the external medium was characterized by the equilibrium parameters of vinyl plastic: $E = 0.03 \times 10^{-5}$ MPa, $\nu = 0.354$ and $\rho = 1.3$ t/m^3 or plexiglass: $E = 0.525 \times 10^{-4}$ MPa, $\nu = 0.35$ and $\rho = 1.18$ t/m^3, while the inclusion was characterized by the parameters of steel: $E_0 = 19 \times 10^{-4}$ MPa, $\nu = 0.3$ and $\rho = 8$ t/m^3. The thickness of the inclusion was taken to be $0.09a$. Note that a steel inclusion in vinyl plastic corresponds to a value of $\kappa = 3$ while in plexiglass it corresponds to $\kappa = 1$.

In Fig. 1 we show curves of $\sigma^* = \sigma(\theta_0)/(2a)$ as a function of the wave parameter x. Curves 1 and 2 correspond to an absolutely rigid inclusion ($Z = \infty$) while curves 3 and 4 correspond to an elastic inclusion, situated in vinyl plastic, with $\theta_0 = 0$ and $\theta_0 = \pi/2$, respectively. It follows from curves 2 and 4 in the figure that when $x > 1$ the inclusion behaves as an absolutely rigid body, i.e. effects related to its longitudinal oscillations are negligibly small. These effects are important only in the Rayleigh region, where the oscillation frequencies are close to the natural frequency $k_l \cong 0$, i.e. $k_l a \ll x$ ($\xi_1^{-1} = 0.038$). At the same time, it follows from curves 1 and 3 that $\sigma(0)$ when $x > 1$ is fairly accurately determined by its Kirchhoff approximation $4aZ^2/(1 + Z^2)$.

In Fig. 2 we show graphs of $f^* = F(\theta_0)/2$ for vinyl plastic (a) and plexiglass (b) for $x = 10$. Curves 1 correspond to longitudinal waves ($A + L$) and curves 2 correspond to transverse waves ($A = T$). We know [8], that for inclined incidence of a sound wave on an elastic plate situated in a liquid, over a range of certain angles intense reflection is observed in a direction opposite to the direction of the incident wave (so-called non-specular reflection, due to flexural and longitudinal waves in the plate). The non-specular reflection of longitudinal waves, due to flexural waves of a thin-walled inclusion, can be clearly seen in Fig. 2(a) in the angular range $\theta_* \cong 23°$ ($\sin\theta_* \cong \xi_3^{-1} = 0.383$). Non-specular reflection of longitudinal waves, due to longitudinal waves in the inclusion (Fig. 2b), are less pronounced and occur in the range $\theta_* \cong 30°$ ($\sin\theta_* \cong \xi_1^{-1} = 0.523$). These resonances are fairly wide here unlike the resonances that occur in the oscillations of elastic plates in a liquid.

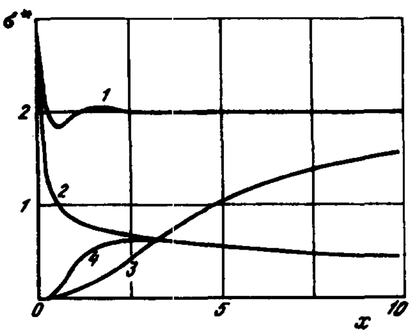

Fig. 1.

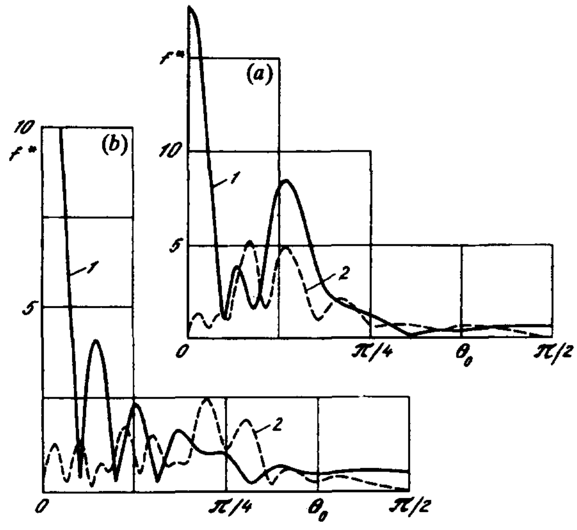

Fig. 2.

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